Analysis of a Transmission Cavity Wavemeter*

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Summary—A section of transmission line partially closed off at each end constitutes a cavity wavemeter. If fixed in length, it may be used as a reference cavity; or if tunable, it may be used to determine frequency. Such a cavity is here treated systematically as a lossy transmission line, with the two end couplings either lossless or symmetrical. The analysis is by means of the transfer or wave matrix. Various expressions are derived which have previously not been obtained, or for which only approximate expressions have been derived from "equivalent circuits."

I. Introduction

A "transmission wavemeter" is a wavemeter which normally reflects most of the power incident on the generator side. It transmits appreciable power from the generator to the load only over a narrow frequency band. The load is often a crystal detector which peaks sharply as the wavemeter or generator is tuned through resonance. This transmission line circuit may be represented as shown in Fig. 1, where

a_1, a_2, a_2', and a_3 are wave amplitudes in the direction from generator to load (left to right in Fig. 1) at the reference planes shown,

b_1, b_2, b_2', and b_3 are wave amplitudes in the reverse direction (right to left in Fig. 1) at the same reference planes, and

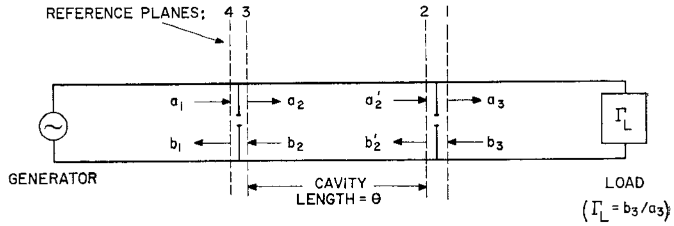
$$\Gamma_L = \frac{b_3}{a_3} \tag{1}$$

is the reflection coefficient of the load as measured at the last reference plane.

The wave amplitudes a and b will be defined in terms of power flow[1] by

$$|a|^2 = \text{power flow in the forward direction (i.e. towards the load)},$$
$$|b|^2 = \text{power flow in the reverse direction (i.e. back towards the generator)}. \tag{2}$$

The cavity is supposed to be coupled to the outside by two identical couplings (holes or irises, etc.), one at each end of the cavity (Fig. 1). We shall also take each coupling in itself as *either* lossless *or* symmetrical (and possibly lossy), since this simplifies the analysis, and the general case of unsymmetrical lossy couplings is not commonly found. No further restriction is placed on the kind of coupling (*e.g.* iris thickness, etc.).



TRANSFER MATRICES BETWEEN REFERENCE PLANES:
T(1→2) = T, T(2→3) = Θ, T(3→4) = T.

Fig. 1—Transmission wavemeter cavity.

The electrical length of the cavity, θ radians, is measured between the two reference planes inside the cavity, and is defined by

$$\theta = \frac{2\pi \times \text{cavity length (between inside reference planes)}}{\text{guide wavelength, } \lambda_g}. \tag{3}$$

The treatment presented here is exact to the extent that a) higher-order modes may be neglected, and b) the variable quantity in the definition of θ is cavity length rather than guide wavelength (cavity tuned at fixed frequency), since the coupling parameters at each end of the cavity will generally be frequency sensitive and so vary with guide wavelength. Under certain conditions it is not necessary to stipulate b), and θ can be taken as a frequency variable (fixed cavity and tunable signal generator). This occurs for instance when the couplings are ideal transformers,[2] which can be realized closely in practice with E-plane waveguide steps.

Resonance

In a simple series L-C-R circuit driven by a constant voltage generator, the current in the circuit, the voltage across L, the charge on C, etc., reach maximum or "resonance" at slightly different frequencies. The "resonant frequency" is usually understood to be the frequency for current resonance, and is independent of the resistance R.

With a cavity too, the reflected wave, the transmitted wave, and the internal fields reach maximum or resonance at slightly different values of θ. However, Q is generally large; it is usually several thousand in microwave wavemeters, so that to speak of a resonant length or frequency is in general substantially correct.

Of all the resonant lengths, the most convenient one is perhaps that length which gives amplitude resonance,

* Received by the PGMTT, February 25, 1960; revised manuscript received, March 11, 1960.

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[1] L. Young, "Transformation matrices," IRE Trans. on Circuit Theory, vol. CT-5, pp. 147–148; June, 1958.

[2] L. Young, "Design of Microwave Stepped Transformers with Applications to Filters," Ph.D. dissertation, The Johns Hopkins University, Baltimore, Md.; 1959.

that is, maximum wave amplitudes inside the cavity. There are two reasons for this choice. The first is that it is applicable to every cavity wavemeter, regardless of external connections. The second reason is that the pole, which determines the maximum internal cavity fields, also occurs as a pole in the expressions for the reflected and transmitted amplitudes, as will be seen later.

II. Analysis by Transfer Matrices

Consider a wavemeter symmetrically coupled as shown in Fig. 1 by two separate couplings, into a wave generator on the left and into a load on the right. (The generator and load are only coupled via the cavity.)

The wave or transfer matrix[1–6] will be used. The cavity length θ is introduced by the line transfer matrix $\mathbf{\Theta}$, given by

$$\mathbf{\Theta} = \begin{pmatrix} e^{J\theta} & 0 \\ 0 & e^{-J\theta} \end{pmatrix}, \tag{4}$$

where

$$J = j + \frac{A}{2\pi}, \tag{5}$$

A being the attenuation of the cavity transmission line in nepers per guide wavelength. (For a wavemeter with no internal losses, $A=0$, and then J reduces to $j=\sqrt{-1}$.)

Since each coupling hole has been taken to be *either* lossless *or* symmetrical (and possibly lossy), the two reference planes associated with each coupling may be chosen so that the reflection coefficient Γ of a single coupling hole is the same from either side (in amplitude as well as phase), and its transmission coefficient T is the same in both directions. Then the transfer matrix of a single coupling hole may be written[5]

$$\mathbf{T} = \frac{1}{T}\begin{pmatrix} 1 & -\Gamma \\ \Gamma & T^2 - \Gamma^2 \end{pmatrix}. \tag{6}$$

From Fig. 1,

$$\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \mathbf{T}^{-1}\mathbf{\Theta}^{-1}\mathbf{T}^{-1}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}. \tag{7}$$

In full,

$$\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \frac{1}{T^2}\begin{pmatrix} T^2 - \Gamma^2 & \Gamma \\ -\Gamma & 1 \end{pmatrix}\begin{pmatrix} e^{-J\theta} & 0 \\ 0 & e^{J\theta} \end{pmatrix} \cdot \begin{pmatrix} T^2 - \Gamma^2 & \Gamma \\ -\Gamma & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}. \tag{8}$$

[3] G. L. Ragan, "Microwave Transmission Circuits," Mass. Inst. Tech. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 9, pp. 551–554; 1948.

[4] C. G. Montgomery, R. H. Dicke and E. M. Purcell, "Principles of Microwave Circuits," Mass. Inst. Tech. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 8, pp. 150–151; 1948.

[5] L. Young, "An Analysis of Resonant Cavities by Matrix Methods," M.S. thesis, The Johns Hopkins University, Baltimore, Md.; 1955.

[6] L. Young, "Branch guide directional couplers," *Proc. Natl. Electronics Conf.*, Chicago, Ill., vol. 12, pp. 723–732; 1956.

Now b_3/a_3 is the reflection coefficient Γ_L of the load in the reference plane of the cavity, on the load side:

$$\Gamma_L = \frac{b_3}{a_3}. \tag{1}$$

Eq. (8) represents two simultaneous equations. Substituting (1) and eliminating b_1, yields

$$\frac{a_3}{a_1} = \frac{T^2}{D_1} \tag{9}$$

where

$$D_1 = (1 - \Gamma_L\Gamma)e^{J\theta} - \Gamma\{\Gamma + \Gamma_L(T^2 - \Gamma^2)\}e^{-J\theta}. \tag{10a}$$

Dividing the two simultaneous equations (8), substituting (1), and solving for b_1/a_1 yields

$$\frac{b_1}{a_1} = \frac{\Gamma(1-\Gamma_L\Gamma)e^{J\theta} + (T^2-\Gamma^2)\{\Gamma+\Gamma_L(T^2-\Gamma^2)\}e^{-J\theta}}{D_1}. \tag{11}$$

From (11) and from

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix},$$

it follows that

$$\frac{a_2}{a_1} = \frac{T(1 - \Gamma_L\Gamma)e^{J\theta}}{D_1} \tag{12}$$

and

$$\frac{b_2}{a_1} = \frac{T\{\Gamma + \Gamma_L(T^2 - \Gamma^2)\}e^{-J\theta}}{D_1}. \tag{13}$$

III. Analysis for Lossless Couplings

From here to the end of the paper it will be supposed that the couplings themselves introduce no loss. Then from energy considerations

$$|\Gamma|^2 + |T|^2 = 1. \tag{14}$$

The reference planes of the lossless coupling hole may be chosen (without loss in generality) so that Γ is real; then it can be shown[2,5] that T must be imaginary, and therefore (14) becomes

$$\Gamma^2 - T^2 = 1. \tag{15}$$

Eqs. (10) to (13) then reduce to

$$\frac{a_3}{a_1} = \frac{T^2 e^{-J\theta}}{D_2} \tag{16}$$

$$\frac{b_1}{a_1} = \frac{\Gamma(1 - \Gamma_L\Gamma) - (\Gamma - \Gamma_L)e^{-2J\theta}}{D_2}, \tag{17}$$

$$\frac{a_2}{a_1} = \frac{T(1 - \Gamma_L\Gamma)}{D_2}, \tag{18}$$

$$\frac{b_2}{a_1} = \frac{T(\Gamma - \Gamma_L)e^{-2J\theta}}{D_2} \tag{19}$$

where

$$D_2 = (1 - \Gamma_L\Gamma) - \Gamma(\Gamma - \Gamma_L)e^{-2J\theta}. \qquad (10b)$$

Reflectionless Load

If further $\Gamma_L = 0$ (for a reflectionless load), then

$$\frac{a_3}{a_1} = \frac{T^2 e^{-2J\theta}}{D_3}, \qquad (20)$$

$$\frac{b_1}{a_1} = \frac{\Gamma(1 - e^{-2J\theta})}{D_3} \qquad (21)$$

$$\frac{a_2}{a_1} = \frac{T}{D_3}, \qquad (22)$$

$$\frac{b_2}{a_1} = \frac{T\Gamma e^{-2J\theta}}{D_3} \qquad (23)$$

where

$$D_3 = 1 - \Gamma^2 e^{-2J\theta}. \qquad (10c)$$

Poles and Zeros

The response functions (9) to (13) have the same pole in the complex θ plane, given by

$$e^{2J\theta} = \frac{\Gamma[\Gamma + \Gamma_L(T^2 - \Gamma^2)]}{1 - \Gamma_L\Gamma}. \qquad (24)$$

In the special case of lossless coupling (Γ=real, with T=imaginary), reflectionless load ($\Gamma_L = 0$), and zero cavity waveguide attenuation ($A = 0$), the resonant length is

$$\theta_0 = n\pi, \qquad (n = \text{integer}), \qquad (25)$$

when it is measured between reference planes defined by Γ=real. The multiplier n will be called the Harmonic Number. If the losses are small, n is still nearly integral. Then

$$n = [n] \pm \epsilon, \qquad (26)$$

where $[n]$ is integral and ϵ is small. In this case $[n]$ will be called the Harmonic Number.

The only zero occurs in the expressions for the reflected wave b_1/a_1. It is given by

$$e^{2J\theta} = \frac{(T^2 - \Gamma^2)\{\Gamma + \Gamma_L(T^2 - \Gamma^2)\}}{\Gamma(1 - \Gamma_L\Gamma)}. \qquad (27)$$

This zero will generally be close to the pole. If the cavity line attenuation $A = 0$, the transmitted wave and fields inside the cavity reach maximum or minimum values together; the reflected wave will do so in general at a slightly different θ. This value may be found graphically by plotting the locus of the vector b_1/a_1 for various θ near resonance.

Resonant Length or Frequency Pulling by a Mismatched Load

Eqs. (24) and (25) establish the resonant length pulling due to a mismatched load when the cavity coupling is lossless. Then θ has to be increased by

$$\delta\theta = \frac{1}{2}\arg\left[\frac{\Gamma(\Gamma - \Gamma_L)}{1 - \Gamma_L\Gamma}\cdot e^{-A\theta/\pi}\right] \qquad (28)$$

$$= \frac{1}{2}\arg\left[\frac{\Gamma(\Gamma - \Gamma_L)}{1 - \Gamma_L\Gamma}\cdot e^{-nA}\right], \qquad (29)$$

if it is not too far from the resonant value $\theta = n\pi$.

Note that if both Γ and Γ_L are real in the same reference plane, there will be no pulling.

Maximum Transmitted Amplitude

The maximum values of the internal and transmitted waves are obtained when $\theta = n\pi$ is substituted in (20), (22), and (23), if the load is reflectionless, or θ from (29), if it is not, into (16), (18), and (19).

We thus obtain for a reflectionless load ($\Gamma_L = 0$),

$$\left|\frac{a_3}{a_1}\right|_{\max} = \frac{|T|^2}{1 - |\Gamma|^2 m^2} = \frac{1 - |\Gamma|^2}{1 - |\Gamma|^2 m^2} \qquad (30)$$

where

$$m^2 = e^{-A\theta/\pi}, \qquad (31)$$

while if the load is not reflectionless

$$\left|\frac{a_3}{a_1}\right|_{\max} = \frac{1 - |\Gamma|^2}{|1 - \Gamma_L\Gamma| - |\Gamma(\Gamma - \Gamma_L)|\, m^2}. \qquad (32)$$

Eq. (30) can be written in terms of Q. Define

$$s = \frac{\lambda_g}{\lambda} \qquad (33)$$

where λ_g=guide wavelength, and λ=free space wavelength. The quantity s^2 often occurs in the theory of dispersive transmission lines[4-6] and will be called the "dispersion factor."

It can be shown[7] that Q is given very closely by

$$Q = \frac{\pi s^2}{A}\cdot\frac{1 - e^{-A\theta/\pi}}{1 - |\Gamma|^2 e^{-A\theta/\pi}}. \qquad (34)$$

Since

$$1 - |\Gamma|^2 m^2 \doteqdot \frac{n\pi s^2}{Q}, \qquad (35)$$

therefore

$$\left|\frac{a_3}{a_1}\right|_{\max} \doteqdot \frac{|T|^2 Q}{n\pi s^2}. \qquad (36)$$

Resonance Amplification

If Γ_L is not zero, the maximum amplitude inside the cavity is given by $G|T|$, where G is the "resonance amplification,"

$$G|T| \doteqdot \left.\frac{|a_2| + |b_2|}{|a_1|}\right|_{\max}$$

$$= \frac{|T|\{|1 - \Gamma_L\Gamma| + |\Gamma - \Gamma_L|\, m^2\}}{|1 - \Gamma_L\Gamma| - |\Gamma(\Gamma - \Gamma_L)m^2}, \qquad (37)$$

[7] L. Young, "Q-factors of a transmission line cavity," IRE TRANS. ON CIRCUIT THEORY, vol. CT-4, pp. 3–5; March, 1957.

while if $\Gamma_L = 0$, this reduces to

$$\left. \frac{|a_2| + |b_2|}{|a_1|} \right|_{\max} = \frac{|T|(1 + |\Gamma| m^2)}{1 - |\Gamma|^2 m^2} \cdot \quad (38)$$

For the rest of this discussion consider only the case of a reflectionless load, $\Gamma_L = 0$.

Since $|\Gamma|^2 \doteqdot 1$ and $m^2 \doteqdot 1$,

$$G|T| \doteqdot \left. \frac{|a_2| + |b_2|}{|a_1|} \right|_{\max} \doteqdot \frac{2|T|}{1 - |\Gamma|^2 m^2} \doteqdot \frac{2Q|T|}{n\pi s^2}, \quad (39)$$

Thus the resonance amplification G is approximately

$$G \doteqdot \frac{2Q}{n\pi s^2} \cdot \quad (40)$$

In series (or shunt) L-C-R circuits the voltage (or current) resonance amplification is Q. The resonance amplification G here defined is proportional to Q, but if expressed in terms of Q, is also inversely proportional to the harmonic number n, and the dispersion factors s^2.

Bandwidth

The cavity bandwidth can be obtained from (20) or (22), when $\Gamma_L = 0$. The bandwidth is defined in the usual way by $W = 2\Delta f$, where $\pm\Delta f$ is the deviation from the resonant frequency, which reduces

$$\left| \frac{a_2}{a_1} \right| \text{ or } \left| \frac{a_3}{a_1} \right| \text{ to } \frac{1}{\sqrt{2}} \text{ of their maximum values.}$$

Now

$$df = -f\frac{d\lambda}{\lambda} = -\frac{f}{s^2}\frac{d\lambda_g}{\lambda_g} \cdot \quad (41)$$

Hence

$$W = 2\Delta f \doteqdot \frac{2f_0}{s^2}\frac{\Delta\theta}{\theta_0} = \frac{2f_0}{s^2}\frac{\Delta\theta}{n\pi}, \quad (42)$$

where $\pm\Delta\theta$ is the deviation of θ from its resonant value $\theta_0 = n\pi$, which reduces $|a_2/a_1|$ and $|a_3/a_1|$ to $1/\sqrt{2}$ of their maximum values.

For high Q this will be given approximately by

$$1 - |\Gamma|^2 m^2 \doteqdot 2\Delta\theta. \quad (43)$$

Hence by (35),

$$2\Delta Q \doteqdot \frac{n\pi s^2}{Q} \cdot \quad (44)$$

Therefore

$$W \doteqdot \frac{f_0}{Q} \cdot \quad (45)$$

This result is not surprising since it is the familiar expression for a series or shunt L-C-R circuit, except that there it is exact (whereas for a transmission line cavity it is only a good approximation for high Q).

IV. Conclusion

An analysis of a transmission wavemeter has been presented which, given a single-mode in each section of transmission line, is exact. This treatment is based on the transfer matrix, and does not require the use of equivalent L-C-R circuits.[8]

Acknowledgment

This work is based largely on a Master's essay written at The Johns Hopkins University, Baltimore, Md., some years ago, with help and advice from Dr. W. H. Huggins and Dr. D. D. King.

[8] C. G. Montgomery, "Technique of Microwave Measurements," Mass. Inst. Tech. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 11, ch. 5; 1947.